Stochastic flows approach to Dupire's formula

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Abstract

The probabilistic equivalent formulation of Dupire's PDE is the Put-Call duality equality. In local volatility models including exponential Lévy jumps, we give a direct probabilistic proof for this result based on stochastic flows arguments. This approach also enables us to check the probabilistic equivalent formulation of various generalizations of Dupire's PDE recently obtained by Pironneau [7] by the adjoint equation technique in the case of complex options.

Introduction

The second order derivative of the price of a Call option with respect to the strike variable is equal to the risk-neutral density of the underlying stock at maturity multiplied by the actualization factor. In a stock model with a local volatility function and a proportional dividend rate ((0.1) with $\mu = m = 0$), Dupire [4] takes advantage of this specificity to obtain a PDE (see (2.3) for $m = 0$) satisfied by the Call pricing function in the maturity and strike variables. His proof consists in integrating twice in space the Fokker-Planck equation governing the time evolution of the density of the stock price. Alternatively, one may use the Green function of the problem or the adjoint equation technique [7]. For calibration purposes, Dupire's PDE permits to express the local volatility function in terms of the function giving the Call prices for all strikes and maturities.

Dupire's PDE can be interpretated as the pricing PDE for a Put option. This leads to the Put-Call duality (equality (2.2) for $\tilde{\mu} = \mu = m = 0$) which is in fact an equivalent formulation : the Call price is transformed into the Put price by simultaneous exchange of the interest and dividend rates and of the spot and strike prices in addition to time-reversal of the local volatility function. To our knowledge, no direct probabilistic proof is available for the equality of the expectations giving the Call and Put prices. In [2], in models including exponential Lévy jumps, Carr and Andreasen derive a PIDE generalizing Dupire's PDE by computing the evolution of the Call payoff with respect to maturity thanks to the Itô-Tanaka formula and taking expectations. The present paper deals with such models (see (0.1)). In the second section, we give a probabilistic proof of the Put-Call duality (2.2) equality equivalent to this PIDE. We check the equality of the expectations by an argument based on stochastic flows of diffeomorphisms. The flow properties of the SDE (0.1) involved in this argument are introduced in the first section and proved in the appendix.

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In a recent paper, Pironneau [7] obtains various generalizations of Dupire's PDE to complex options by the adjoint equation technique. More precisely, for a given complex option, he shows that it is possible to compute the pricing function for all strikes and maturities by solving a single PDE. In calibration procedures, solving this PDE instead of one pricing PDE for the maturity and strike of each quoted option permits important computation time reduction. Most of these generalized Dupire's PDEs have equivalent probabilistic interpretations similar to the Put-Call duality. In the third and fourth sections of the paper, we use stochastic flows to check the equivalent interpretations corresponding to binary and options written on two assets.

The fifth section deals with barrier options in local volatility models without jumps. In section 1.1 [7], Pironneau addresses two-barriers options but we have only been able to give a probabilistic equivalent interpretation (see (5.1)) in the one-barrier case. Moreover, besides particular cases, it seems challenging to give a probabilistic proof of this equivalent formulation. The case of American options is not addressed in [7]. In [1], we deal with the case of perpetual options when the local volatility function does not depend on time. For the perpetual American Call price to be equal to the perpetual American Put price, in addition to the exchanges of the interest and dividend rates and of the spot and strike prices, the volatility function has to be modified. Our approach consists in deriving and studying an ODE satisfied by the exercise boundary as a function of the strike variable. Again, a direct probabilistic proof of the duality results appears challenging. The stochastic flow approach presented in the present paper does not seem suited to deal with options involving stopping times like barrier and American options.

Notations : For $T > 0$ and m a measure on $\mathbb{R}$ such that $\int_{\mathbb{R}}(1+e^l) \wedge l^2 m(dl) < +\infty$, let $(W_t)_{t\in[0,T]}$ be a standard Brownian motion and μ denote an independent Poisson random point measure on $(0,T]\times\mathbb{R}$ with intensity $m(dl)dt$.
We consider the following risk-neutral evolution for the underlying stock price

$$dX_t^x = \sigma(t, X_t^x)X_t^x dW_t + (r-\delta)X_t^x dt + X_{t^-}^x \int_{\mathbb{R}} (e^l - 1)(\mu(dt, dl) - m(dl)dt),\ X_0^x = x > 0 \quad (0.1)$$

where r denotes the interest rate and δ the dividend rate. The local volatility function $\sigma(t,x)$ is assumed to belong to the space

$$\mathcal{V} = \left\{ f : [0,T]\times(0,+\infty) \to \mathbb{R} : \sup_{[0,T]\times\mathbb{R}} \sum_{k=0}^{3} |x^k \partial_2^k f(t,x)| < +\infty \right\}$$

where $\partial_2^k f$ denotes the k-th order derivative of the function f with respect to its second variable. The process $(\hat{W}_t = W_{T-t} - W_T)_{t\in[0,T]}$ obtained by time-reversal of W is a Brownian motion independent from the image $\hat{\mu}$ of μ by the mapping $(t,l) \in (0,T]\times\mathbb{R} \to (T-t, -l)$ which is a Poisson random point measure on $[0,T)\times\mathbb{R}$ with intensity $\hat{m}(dl)dt$ where $\hat{m}$ denotes the image of m by $l \in \mathbb{R} \to -l$.
Let us also introduce the Lévy processes

$$L_t = \int_{(0,t]\times\mathbb{R}} l(\mu(ds,dl) - 1_{\{|l|\le 1\}} m(dl)ds) - t\int_{\mathbb{R}} (e^l - 1 - l1_{\{|l|\le1\}})m(dl)$$

$$\text{and } \hat{L}_t = L_{T-t} - L_T = \int_{[0,t)\times\mathbb{R}} l(\hat{\mu}(ds,dl) - 1_{\{|l|\le 1\}}\hat{m}(dl)ds) + t\int_{\mathbb{R}} (e^l - 1 - l1_{\{|l|\le1\}})m(dl). \quad (0.2)$$

By the Lévy-Kinchine formula,

$$\mathbb{E}(e^{iuL_t}) = e^{t\psi(u)} \text{ where } \psi(u) = -iu\int_{\mathbb{R}}(e^l - 1 - l1_{\{|l|\leq 1\}})m(dl) + \int_{\mathbb{R}}(e^{iul} - 1 - iul1_{\{|l|\leq 1\}})m(dl)$$
$$= \int_{\mathbb{R}}(e^{iul} - 1 - iu(e^l - 1))m(dl). \tag{0.3}$$

1 Stochastic flows of diffeomorphisms

According to the theory of stochastic flows of diffeomorphisms developped by Kunita [6], almost surely, the solution at time $t > 0$ of a Stochastic Differential Equation with regular coefficients is a diffeomorphism as a function of the initial position. Derivatives of the solution with respect to the initial condition solve the linear equations obtained by formal derivation of the SDE. Last, the inverse diffeomorphism is associated with a backward SDE. In the following proposition, we adapt these results to a slight generalization of the SDE with jumps preserving positivity (0.1) considered in the present paper.

Proposition 1.1 *Assume that $\sigma, \beta \in \mathcal{V}$ and let $\eta(t,x) = x\sigma(t,x)$. Then trajectorial uniqueness holds for the stochastic differential equations*

$$dX_t^x = \eta(t, X_t^x)dW_t + \beta(t, X_t^x)X_t^x dt + X_{t-}^x\int_{\mathbb{R}}(e^l - 1)(\mu(dt,dl) - m(dl)dt),\ t \leq T,\ X_0^x = x > 0$$
$$dZ_t^z = \eta(T-t, Z_t^z)d\hat{W}_t + Z_t^z(\sigma\partial_2\eta - \beta)(T-t, Z_t^z)dt + Z_{t-}^z\int_{\mathbb{R}}(e^l - 1)(\hat{\mu}(dt,dl) + m(dl)dt),\ t \leq T$$

where $Z_0^z = z > 0$ and $\int_{\mathbb{R}}(e^l - 1)(\hat{\mu}(dt,dl) + m(dl)dt)$ stands for

$$\int_{\mathbb{R}}(e^l - 1)(\hat{\mu}(dt,dl) - 1_{|l|\leq 1}\hat{m}(dl)dt) + \int_{\mathbb{R}}(e^l - 1 + 1_{\{|l|\leq 1\}}(e^{-l} - 1))m(dl).$$

They admit solutions such that for almost all $\omega \in \Omega$, the mappings $x \to X_T^x$ and $z \to Z_T^z$ are inverse increasing diffeomorphisms of $(0, +\infty)$,

$$\forall t \in [0,T],\ Z_t^z = ze^{\int_0^t \sigma(T-s,Z_s^z)d\hat{W}_s + \int_0^t[\sigma\partial_2\eta - \beta - \frac{\sigma^2}{2}](T-s,Z_s^z)ds + \hat{L}_{t+}} \text{ (with } \hat{L}_{T+} = \hat{L}_T\text{)}, \tag{1.1}$$
$$\partial_x X_T^x = e^{\int_0^T \partial_2\eta(s,X_s^x)dW_s + \int_0^T(\beta + X_s^x\partial_2\beta - \frac{(\partial_2\eta)^2}{2})(s,X_s^x)ds + L_T} \tag{1.2}$$
$$\text{and } \forall x,z > 0,\ \{X_T^x \geq z\} = \{x \geq Z_T^z\}. \tag{1.3}$$

The rather technical proof of this proposition is postponed to the appendix. To deduce the Put-Call duality equality (2.2), we are going to check the equality of the derivatives of both sides with respect to x. The next result enables us to justify the formula $\partial_x\mathbb{E}(e^{-rT}(X_T^x - y)^+) = \mathbb{E}\left(e^{-rT}\partial_x X_T^x 1_{\{X_T^x \geq y\}}\right)$ obtained by formal derivation and where the indicator funtion in the right-hand side will be replaced thanks to (1.3). Its proof is also postponed to the appendix.

Proposition 1.2 *Under the assumptions and notations of Proposition 1.1, when for some $z > 0$, the local volatility function σ does not vanish on a neighbourhood of (T,z) in $[0,T] \times (0,+\infty)$, then*

$$\forall x > 0,\ \mathbb{P}(X_T^x = z) = \mathbb{P}(Z_T^z = x) = 0. \tag{1.4}$$

Last, if $\beta(t,x)=\gamma$ for some constant $\gamma\in\mathbb{R}$ then

$$\forall x>0,\ \mathbb{E}(e^{-\gamma T}X_T^x)=x \text{ and } \mathbb{E}(e^{-\gamma T}\partial_x X_T^x)=1 \tag{1.5}$$

and for any sequence $(h_n)_{n\geq 0}$ of non-zero real numbers greater than $-x$ converging to zero, the random variables $\left((X_T^{x+h_n}-X_T^x)/h_n\right)_{n\geq 0}$ are uniformly integrable.

2 Standard options

For $y>0$, let $C(T,x,y)=\mathbb{E}(e^{-rT}(X_T^x-y)^+)$ denote the price of the Call option with maturity T and strike y written on the underlying X^x evolving according to (0.1).
We are going to derive the Put-Call duality (2.2) from the following proposition.

Theorem 2.1 *Assume that the local volatility function does not vanish in a neighbourhood of (T,y) in $[0,T]\times(0,+\infty)$. Then*

$$\forall x>0,\ \partial_x C(T,x,y)=\partial_x\mathbb{E}\left(e^{-\delta T}(x-Y_T^y)^+\right) \tag{2.1}$$

where $dY_t^y=\sigma(T-t,Y_t^y)Y_t^y d\hat{W}_t+(\delta-r)Y_t^y dt+Y_{t^-}^y\int_{\mathbb{R}}(e^{-l}-1)(\tilde{\mu}(dt,dl)-e^l m(dl)dt),\ Y_0^y=y$ and $\tilde{\mu}$ is a Poisson random point measure with intensity $e^l m(dl)dt$ independent from $\hat{W}$.

As $\mathbb{E}\left(e^{-\delta T}(x-Y_T^y)^+\right)\leq e^{-\delta T}x$ and $C(T,x,y)\leq\mathbb{E}(e^{-rT}X_T^x)$ with $\mathbb{E}(e^{-rT}X_T^x)=e^{-\delta T}x$ according to (1.5), one has $\lim_{x\to 0^+}C(T,x,y)=\lim_{x\to 0^+}\mathbb{E}\left(e^{-\delta T}(x-Y_T^y)^+\right)=0$. Hence the Put-Call duality follows from (2.1) :

Corollary 2.2 *If the local volatility function does not vanish in a neighbourhood of (T,y),*

$$\forall x>0,\ C(T,x,y)=\mathbb{E}\left(e^{-\delta T}(x-Y_T^y)^+\right). \tag{2.2}$$

Remark 2.3 *If the local volatility function σ is positive and belongs to $\mathcal{V}$ for any $T>0$, then the Put-Call duality (2.2) holds for all $(T,y)\in[0,+\infty)\times(0,+\infty)$. This implies Dupire's PIDE. Indeed, for $s\in[0,T]$, let $(Y_t^{s,y,T})_{t\in[s,T]}$ solve $Y_s^{s,y,T}=y$ and for $t\in[s,T]$,*

$$dY_t^{s,y,T}=\sigma(T-t,Y_t^{s,y,T})Y_t^{s,y,T}d\hat{W}_t+(\delta-r)Y_t^{s,y,T}dt+Y_{t^-}^{s,y,T}\int_{\mathbb{R}}(e^{-l}-1)(\tilde{\mu}(dt,dl)-e^l m(dl)dt).$$

For fixed s, by time translation, the expectation $\mathbb{E}\left(e^{-\delta(T-(T-s))}(x-Y_T^{T-s,y,T})^+\right)$ does not depend on T greater than s and may be denoted $u(s,y)$. By the Feynman-Kac formula, one has

$$\begin{cases}\partial_s u(s,y)=\frac{1}{2}\sigma^2(s,y)y^2\partial_{yy}u(s,y)+(\delta-r)y\partial_y u(s,y)-\delta u(s,y)\\ \qquad\qquad +\int_{\mathbb{R}}\left[u(s,ye^{-l})-u(s,y)-\partial_y u(s,y)y(e^{-l}-1)\right]e^l m(dl),\ s,y>0\\ u(0,y)=(x-y)^+,\ y>0\end{cases}.$$

Since $C(T,x,y)=u(T,y)$, one deduces that the function C solves Dupire's PIDE in the variables (T,y) :

$$\begin{cases}\partial_T C(T,x,y)=\frac{1}{2}\sigma^2(T,y)y^2\partial_{yy}C(T,x,y)+(\delta-r)y\partial_y C(T,x,y)-\delta C(T,x,y)\\ \qquad\qquad +\int_{\mathbb{R}}\left[C(T,x,ye^{-l})-C(T,x,y)-\partial_y C(T,x,y)y(e^{-l}-1)\right]e^l m(dl),\ T,y>0\\ C(0,x,y)=(x-y)^+,\ y>0\end{cases}. \tag{2.3}$$

Conversely, if (2.3) holds, the function $v(t,y) = C(T-t,x,y)$ satisfies the PIDE

$$\partial_t v(t,y) + \frac{1}{2}\sigma^2(T-t,y)y^2\partial_{yy}v + (\delta - r)y\partial_y v - \delta v$$
$$+ \int_{\mathbb{R}} \left[v(t,ye^{-l}) - v(t,y) - \partial_y v(t,y)y(e^{-l}-1)\right] e^l m(dl) = 0$$

with terminal condition $v(T,y) = (y-x)^+$. By the Feynman-Kac representation for the solution of this PIDE, $v(0,y) = \mathbb{E}\left(e^{-\delta T}(x - Y_T^y)^+\right)$ and (2.2) holds.

Proof of Theorem 2.1 : Let $(h_n)_{n\geq 0}$ be a sequence of non-zero real numbers greater than $-x$ converging to zero. Since $x \to X_T^x$ is increasing according to Proposition 1.1, one has

$$0 \leq \frac{(X_T^{x+h_n} - y)^+ - (X_T^x - y)^+}{h_n} \leq \frac{X_T^{x+h_n} - X_T^x}{h_n},$$

and the variables $\left(((X_T^{x+h_n} - y)^+ - (X_T^x - y)^+)/h_n\right)_{n\geq 0}$ are uniformly integrable by Proposition 1.2. By (1.4), these variables converge a.s. to $\partial_x X_T^x 1_{\{X_T^x \geq y\}}$ as $n \to +\infty$. One deduces that $C(T,x,y)$ is differentiable with respect to x and $\partial_x C(T,x,y) = e^{-\delta T}\mathbb{E}\left(e^{(\delta-r)T}\partial_x X_T^x 1_{\{X_T^x \geq y\}}\right)$. By (1.2), this implies

$$\partial_x C(T,x,y) = e^{-\delta T}\mathbb{E}\left(e^{L_T}\mathbb{E}\left(e^{\int_0^T \partial_2\eta(u,X_u^x)dW_u - \frac{1}{2}\int_0^T(\partial_2\eta(u,X_u^x))^2du} 1_{\{X_T^x\geq y\}}\middle|(L_s)_{s\in[0,T]}\right)\right).$$

Since $\partial_2\eta(t,x) = \sigma(t,x) + x\partial_2\sigma(t,x)$ is bounded on $[0,T]\times\mathbb{R}$, for $y=0$ the conditional expectation in the right-hand-side is equal to 1 by Novikov's criterion (Proposition 1.15 p.308 [8]). Therefore, by Girsanov theorem,

$$\mathbb{E}\left(e^{\int_0^T \partial_2\eta(u,X_u^x)dW_u - \frac{1}{2}\int_0^T(\partial_2\eta(u,X_u^x))^2du} 1_{\{X_T^x\geq y\}}\middle|(L_s)_{s\in[0,T]}\right) = \mathbb{E}\left(1_{\{\bar{X}_T^x\geq y\}}\middle|(L_s)_{s\in[0,T]}\right)$$

where $d\bar{X}_t^x = \eta(t,\bar{X}_t^x)dW_t + \beta(t,\bar{X}_t^x)\bar{X}_t^x dt + \bar{X}_{t^-}^x \int_{\mathbb{R}} l(\mu(dl,dt) - 1_{\{|l|\leq 1\}}m(dl)dt),\ \bar{X}_0^x = x$

with $\beta(t,z) = \sigma\partial_2\eta(t,z) + (r-\delta)$. By (1.3), one deduces that

$$\partial_x C(T,x,y) = e^{-\delta T}\mathbb{E}\left(e^{L_T}1_{\{x\geq Z_T^y\}}\right), \tag{2.4}$$

where, according to (1.1) and the definition of β,

$$\forall t\in[0,T],\ Z_t^z = ze^{\int_0^t \sigma(T-s,Z_s^z)d\hat{W}_s + (\delta-r)t - \frac{1}{2}\int_0^t\sigma^2(T-s,Z_s^z)ds + \hat{L}_{t^+}}\ \text{(convention : } \hat{L}_{T^+} = \hat{L}_T).$$

When $m=0$, we are done. Otherwise, we still have to derive the dynamics of $\hat{L}_{t^+}$ under the probability measure with density e^{L_T} with respect to $\mathbb{P}$. By (0.3), for $t\in[0,T]$ and $u\in\mathbb{R}$, $\mathbb{E}(e^{L_T}e^{iu\hat{L}_{t^+}}) = \mathbb{E}(e^{(1-iu)(L_T - L_{T-t})})\mathbb{E}(e^{L_{T-t}}) = e^{t\psi(-(u+i))}$ and

$$\psi(-(u+i)) = \int_{\mathbb{R}}(e^l e^{-iul} - 1 + (iu-1)(e^l-1))m(dl)$$
$$= \int_{\mathbb{R}}(e^{-iul} - 1 + iul1_{\{|l|\leq 1\}})e^l m(dl) + iu\int_{\mathbb{R}}(e^l - 1 - le^l 1_{\{|l|\leq 1\}})m(dl).$$

Therefore, under the probability measure with density e^{L_T} with respect to $\mathbb{P}$,

$$\hat{L}_{t^+} = -\int_{(0,t]\times\mathbb{R}} l(\bar{\mu}(ds,dl) - 1_{\{|l|\leq 1\}}e^l m(dl)ds) + t\int_{\mathbb{R}}(e^l - 1 - le^l 1_{\{|l|\leq 1\}})m(dl)$$

with $\bar{\mu}$ a Poisson random point measure with intensity $e^l m(dl)dt$ independent from W. By Itô's formula

$$dZ_t^z = \eta(T-t, Z_t^z)d\hat{W}_t + (\delta - r)Z_t^z dt + Z_{t^-}^z \int_{\mathbb{R}}(e^{-l} - 1)(\bar{\mu}(dt,dl) - e^l m(dl)dt),\ Z_0^z = z.$$

Since trajectorial and therefore weak uniqueness holds for this SDE with jumps, $\mathbb{E}(e^{L_T} 1_{\{x\geq Z_T^y\}}) = \mathbb{P}(x \geq Y_T^y)$ and by (2.4), $\partial_x C(T,x,y) = e^{-\delta T}\mathbb{E}(1_{\{x\geq Y_T^y\}})$. According to Lebesgue's Theorem, the right-hand-side is equal to $\partial_x \mathbb{E}\left(e^{-\delta T}(x - Y_T^y)^+\right)$ since $\mathbb{P}(Y_T^y = x) = \mathbb{E}(e^{L_T}1_{\{Z_T^y = x\}}) = 0$ by (1.4). ∎

Remark 2.4 *Many authors have obtained another type of Put-Call duality by the following change of numéraire approach :*

$$\mathbb{E}(e^{-rT}(X_T^x - y)^+) = \mathbb{E}^{\mathbb{Q}}\left(e^{-\delta T}\left(x - \frac{yx}{X_T^x}\right)^+\right) \ \textit{with}\ \frac{d\mathbb{Q}}{d\mathbb{P}} = e^{(\delta-r)T}\frac{X_T^x}{x}.$$

In exponential Lévy models, the local volatility function is constant and Fajardo and Mordecki [5] derive (2.2) *by this approach. But in general, the coefficients of the SDE with jumps satisfied by $\frac{xy}{X_t^x}$ under $\mathbb{Q}$ depend on the primal spot variable and dual strike variable x. Then it is not clear to deduce a PIDE in the variables T and y for the pricing function $C(T,x,y)$.*

3 Binary options

For $z > 0$, let $c(T,x,z) = \mathbb{E}\left(e^{-rT}1_{\{X_T^x \geq z\}}\right)$ denote the price of the binary Call option with strike z and maturity T written on the underlying X^x. The following result is a direct consequence of (1.3) :

Proposition 3.1

$$\forall x, z > 0,\ c(T,x,z) = \mathbb{E}\left(e^{-rT}1_{\{x\geq Z_T^z\}}\right) \tag{3.1}$$

where Z^z solves $dZ_t^z = \eta(T-t, Z_t^z)d\hat{W}_t + (\eta\partial_2\eta(T-t,Z_t^z) + (\delta-r)Z_t^z)dt + Z_{t^-}^z\int_{\mathbb{R}}(e^l - 1)(\hat{\mu}(dt,dl) + m(dl)dt),\ Z_0^z = z$.

Remark 3.2 *By a reasoning similar to the one made for the standard Dupire's PIDE in Remark 2.3, one deduces that as a function of the maturity and strike variables, the function $c(T,x,z)$ solves*

$$\begin{cases} \partial_T c(T,x,z) = \frac{1}{2}\sigma^2(T,z)z^2\partial_{zz}c(T,x,z) + (\eta\partial_2\eta(T,z) + (\delta - r)z)\partial_z c(T,x,z) - rc(T,x,z) \\ \qquad\qquad + \int_{\mathbb{R}}\left[c(T,x,ze^{-l}) - c(T,x,z) + \partial_z c(T,x,z)z(e^l - 1)\right]m(dl),\ T,z>0 \\ c(0,x,z) = 1_{\{x\geq z\}},\ z>0 \end{cases}. \tag{3.2}$$

Remarking that

$$\frac{1}{2}\sigma^2(T,z)z^2\partial_{zz}c(T,x,z)+\eta\partial_2\eta(T,z)\partial_z c(T,x,z)=\frac{1}{2}\partial_z\left(\sigma^2(T,z)z^2\partial_z c(T,x,z)\right),$$

one recognizes the PDE obtained in [7] in the absence of jumps (m=0).

Remark 3.3 *The standard Call and the binary Call pricing functions are linked by*

$$C(T,x,y)=\mathbb{E}\left(e^{-rT}\int_y^{+\infty}1_{\{X_T^x\geq z\}}dz\right)=\int_y^{+\infty}c(T,x,z)dz.$$

Integrating the PIDE (3.2) *with respect to z on the interval* $[y,+\infty[$ *and remarking that*

$$\begin{aligned}
&\int_y^{+\infty}\left[c(T,x,ze^{-l})-c(T,x,z)+\partial_z c(T,x,z)z(e^l-1)\right]dz\\
&=\int_{ye^{-l}}^{+\infty}c(T,x,w)e^l dw+(e^l-1)\int_y^{+\infty}\partial_z(zc(T,x,z))dz-e^l\int_y^{+\infty}c(T,x,z)dz\\
&=-\left[e^lC(T,x,ye^{-l})+(e^l-1)y\partial_yC(T,x,y)-e^lC(T,x,y)\right]\\
&=-e^l\left[C(T,x,ye^{-l})-C(T,x,y)-(e^{-l}-1)y\partial_yC(T,x,y)\right]
\end{aligned}$$

one recovers (2.3)*. This alternative proof of* (2.3) *relies on properties of the derivative of the pricing function C with respect to the strike variable, whereas Proposition 2.1 deals with its derivative with respect to the spot variable.*

4 Options written on two assets

We now consider a model with two assets evolving according to the following dynamics

$$\begin{cases}dX_t^{1,x_1}=\sigma_1(t,X_t^{1,x_1})X_t^{1,x_1}dW_t^1+(r-\delta_1)X_t^{1,x_1}dt\\ \qquad\qquad+X_{t^-}^{1,x_1}\int_{\mathbb{R}^2}(e^{l_1}-1)(\mu(dt,dl_1,\mathbb{R})-m(dl_1,\mathbb{R})dt),\ X_0^{1,x_1}=x_1\\ dX_t^{2,x_2}=\sigma_2(t,X_t^{2,x_2})X_t^{2,x_2}dW_t^2+(r-\delta_2)X_t^{2,x_2}dt\\ \qquad\qquad+X_{t^-}^{2,x_2}\int_{\mathbb{R}^2}(e^{l_2}-1)(\mu(dt,\mathbb{R},dl_2)-m(\mathbb{R},dl_2)dt),\ X_0^{2,x_2}=x_2\end{cases},\qquad(4.1)$$

where for $i\in\{1,2\}$, $\sigma_i\in\mathcal{V}$. Here W^1 and W^2 are two standard Brownian motions such that $d<W^1,W^2>_t=\rho_t dt$ with ρ an adapted process and μ is an independent Poisson random point measure on $(0,T]\times\mathbb{R}^2$ with intensity $m(dl_1,dl_2)dt$ where

$$\int_{\mathbb{R}^2}(e^{l_1}+e^{l_2}+1)\wedge(l_1^2+l_2^2)m(dl_1,dl_2)<+\infty.$$

Let for $i\in\{1,2\}$, $\eta_i(t,z_i)=z_i\sigma_i(t,z_i)$, $(\hat{W}_t^i=W_{T-t}^i-W_T^i)_{t\in[0,T]}$ and $\hat{\mu}$ denote the image of μ by the mapping $(t,l_1,l_2)\in(0,T]\times\mathbb{R}^2\to(T-t,-l_1,-l_2)$.

Proposition 4.1 *Let* (Z^{1,z_1},Z^{2,z_2}) *solve*

$$\begin{aligned}
dZ_t^{1,z_1}&=\eta_1(T-t,Z^{1,z_1})d\hat{W}_t^1+(\eta_1\partial_2\eta_1(T-t,Z_t^{1,z_1})+(\delta_1-r)Z_t^{1,z_1})dt\\
&\qquad+Z_{t^-}^{1,z_1}\int_{\mathbb{R}^2}(e^{l_1}-1)(\hat{\mu}(dt,dl_1,\mathbb{R})+m(dl_1,\mathbb{R})dt),\ Z_0^{1,z_1}=z_1\\
dZ_t^{2,z_2}&=\eta_2(T-t,Z^{2,z_2})d\hat{W}_t^2+(\eta_2\partial_2\eta_2(T-t,Z_t^{2,z_2})+(\delta_2-r)Z_t^{2,z_2})dt\\
&\qquad+Z_{t^-}^{2,z_2}\int_{\mathbb{R}^2}(e^{l_2}-1)(\hat{\mu}(dt,\mathbb{R},dl_2)+m(\mathbb{R},dl_2)dt),\ Z_0^{2,z_2}=z_2.
\end{aligned}$$

Then for $w(T,x_1,x_2,z_1,z_2) = \mathbb{E}\left(e^{-rT}1_{\{x_1\geq Z_T^{1,z_1},x_2\geq Z_T^{2,z_2}\}}\right)$, *one has*

$$\begin{aligned}\forall x_1,x_2,y>0, \mathbb{E}\left((X_T^{1,x_1}+X_T^{2,x_2}-y)^+\right) &= \int_0^y w(T,x_1,x_2,z,y-z)dz \\ &+\int_y^{+\infty} w(T,x_1,x_2,z,0)+w(T,x_1,x_2,0,z)dz\end{aligned} \tag{4.2}$$

and $\mathbb{E}\left((X_T^{1,x_1}\vee X_T^{2,x_2}-y)^+\right) = \int_y^{+\infty} w(T,x_1,x_2,z,0)+w(T,x_1,x_2,0,z)-w(T,x_1,x_2,z,z)dz.$

Remark 4.2 *When* $\rho_t = -q(t,X_t^{1,x_1},X_t^{2,x_2})$*, by a reasoning similar to the one made in Remark 3.2, one checks that the function* w *solves the following PIDE obtained in [7] Section 2.1 in the absence of jumps :*

$$\begin{cases}\partial_T w = \sum_{i=1}^2\left[\frac{1}{2}\partial_{z_i}\left(\sigma_i^2(T,z_i)z_i^2\partial_{z_i}w\right)+(\delta_i-r)z_i\partial_{z_i}w\right]-2qz_1z_2\partial_{z_1z_2}w-rw \\ \qquad +\int_{\mathbb{R}^2} w(T,z_1e^{-l_1},z_2e^{-l_2})-[w-\sum_{i=1}^2 z_i(e^{l_i}-1)\partial_{z_i}w](T,z_1,z_2)m(dl_1,dl_2) \\ w(0,x_1,x_2,z_1,z_2)=1_{\{x_1\geq z_1,x_2\geq z_2\}}\end{cases}.$$

Proof : One has

$$\forall y_1,y_2,y\geq 0,\ (y_1+y_2-y)^+ = \int_0^y 1_{\{y_1\geq z,y_2\geq y-z\}}dz+\int_y^{+\infty}1_{\{y_1\geq z\}}+1_{\{y_2\geq z\}}dz.$$

Therefore

$$\mathbb{E}\left((X_T^{1,x_1}+X_T^{2,x_2}-y)^+\right) = e^{-rT}\mathbb{E}\left(\int_0^y 1_{\{X_T^{1,x_1}\geq z,X_T^{2,x_2}\geq y-z\}}dz+\int_y^{+\infty}1_{\{X_T^{1,x_1}\geq z\}}+1_{\{X_T^{2,x_2}\geq z\}}dz\right).$$

Since by (1.3), a.s., $\forall x_1,x_2,z_1,z_2>0,\ \{X_T^{1,x_1}\geq z_1\}=\{x_1\geq Z_T^{1,z_1}\}$ and $\{X_T^{2,x_2}\geq z_2\}=\{x_2\geq Z_T^{2,z_2}\}$,

$$C(T,x_1,x_2,y) = e^{-rT}\mathbb{E}\left(\int_0^y 1_{\{x_1\geq Z_T^{1,z},x_2\geq Z_T^{2,y-z}\}}dz+\int_y^{+\infty}1_{\{x_1\geq Z_T^{1,z}\}}+1_{\{x_2\geq Z_T^{2,z}\}}dz\right)$$

As by Proposition 1.1, $z_2\to Z_T^{2,z_2}$ (resp. $z_1\to Z_T^{1,z_1}$) is an increasing diffeomorphism of $(0,+\infty)$, $\lim_{z_2\to 0^+} w(T,x_1,x_2,z_1,z_2) = e^{-rT}\mathbb{P}(x_1\geq Z_T^{1,z_1})$ (resp. $\lim_{z_1\to 0^+} w(T,x_1,x_2,z_1,z_2) = e^{-rT}\mathbb{P}(x_2\geq Z_T^{2,z_2})$). One easily deduces (4.2).
The formula for the best-off Call option is obtained similarly remarking that

$$\forall y_1,y_2,y\geq 0,\ (y_1\vee y_2-y)^+ = \int_y^{+\infty}1_{\{y_1\geq z\}}+1_{\{y_2\geq z\}}-1_{\{y_1\geq z,y_2\geq z\}}dz.$$

∎

Remark 4.3 *When* σ_i *may depend smoothly on* x_j *for* $j\neq i$ *in (4.1),* $(x_1,x_2)\to(X_T^{1,x_1,x_2},X_T^{2,x_1,x_2})$ *is still a diffeomorphism with inverse* $(z_1,z_2)\to(Z_T^{1,z_1,z_2},Z_T^{2,z_1,z_2})$ *obtained as the solution of a two-dimensional SDE at time* T*. But in general, the events* $\{X_T^{1,x_1,x_2}\geq z_1,X_T^{2,x_1,x_2}\geq z_2\}$ *and* $\{x_1\geq Z_T^{1,z_1,z_2},x_2\geq Z_T^{2,z_1,z_2}\}$ *are not equal and the argument above fails.*

5 Barrier options

In absence of jumps $\boxed{m=0}$ and in the particular case of equal interest and dividend rates $\boxed{r=\delta}$, as a consequence of Proposition 1.2 [7],

$$\forall x, y \geq z > 0,\ \mathbb{E}\left((X_T^x - y)^+ 1_{\{\tau_z^x > T\}}\right) = \mathbb{E}\left((x - Y_{T\wedge t_z^y}^y)^+\right) \tag{5.1}$$

with X^x solving (0.1), Y^y defined in Proposition 2.1, $\tau_z^x = \inf\{t \geq 0 : X_t^x \leq z\}$ and $t_z^y = \inf\{t \geq 0 : Y_t^y \leq z\}$. Notice that since $r = \delta$, there is no drift term in the dynamics of X^x and Y^y and both processes are martingales in their natural filtration. This equality generalizes (2.2) which can be recovered by taking the limit $z \to 0$. It is easy to prove when either x or y is equal to z. Indeed, when $y \geq x = z$, both sides are equal to 0. And when $x \geq y = z$, using the martingale property of X^x for the third equality,

$$\begin{aligned}\mathbb{E}\left((X_T^x - y)^+ 1_{\{\tau_y^x > T\}}\right) &= \mathbb{E}\left((X_T^x - y) 1_{\{\tau_y^x > T\}}\right) = \mathbb{E}(X_T^x) - \mathbb{E}\left(X_T^x 1_{\{\tau_y^x \leq T\}}\right) - y\mathbb{P}(\tau_y^x > T)\\ &= x - y\mathbb{P}(\tau_y^x \leq T) - y\mathbb{P}(\tau_y^x > T)\end{aligned}$$

while the right-hand-side of (5.1) is obviously equal to $x - y$.

Equation (5.1) is equivalent to an equality where x and y play symmetric roles. Indeed substracting (5.1) to (2.2), and using the martingale property of Y^y for the second equality, one gets

$$\begin{aligned}&\mathbb{E}\left((X_T^x - y)^+ 1_{\{\tau_z^x \leq T\}}\right) = \mathbb{E}\left(\left((x - Y_T^y)^+ - (x - z)\right) 1_{\{t_z^y \leq T\}}\right)\\ &\quad= \mathbb{E}\left(\left((x - Y_T^y)^+ - (x - Y_T^y)\right) 1_{\{t_z^y \leq T\}}\right) = \mathbb{E}\left((Y_T^y - x)^+ 1_{\{t_z^y \leq T\}}\right).\end{aligned}$$

In case the local volatility function σ does not depend on the time variable, this last equality obviously holds when $x = y$.
Beside these particular cases, it seems challenging to give a probabilistic proof of (5.1). Derivation of the equality with respect to x or y does not lead to nice probabilistic equalities like (2.1) obtained in the case of standard options. Even in the Black-Scholes model with constant volatility, (5.1) is not obvious. For instance, the change of numéraire approach presented in Remark 2.4 then enables to check that $\mathbb{E}\left((X_T^x - y)^+ 1_{\{\tau_y^x > T\}}\right) = \mathbb{E}\left((x - Y_T^y)^+ 1_{\{\sup_{[0,T]} Y_t^y < \frac{yx}{z}\}}\right)$. But it is not clear that the right-hand-side coincides with the one in (5.1).

Appendix

The proof of Proposition 1.1 relies on the following result concerning SDEs without jumps which is a consequence of Kunita's theory on stochastic flows of diffeomorphisms (see [6] Corollary 4.6.6 for the first statement and equation (21) in the proof of Theorem 4.6.5 for the second).

Proposition 5.1 *Let $a, b : [0,T] \times \mathbb{R} \to \mathbb{R}$ be functions bounded together with their derivatives with respect to their second variable up to the order 3. Then the stochastic differential equation*

$$d\chi_t^\xi = a(t, \chi_t^\xi)dW_t + b(t, \chi_t^\xi)dt,\ t \leq T,\ \chi_0^\xi = \xi \tag{5.2}$$

$$(\textit{resp. } d\zeta_t^\nu = a(T-t, \zeta_t^\nu)d\hat{W}_t + (a\partial_2 a - b)(T-t, \zeta_t^\nu)dt,\ t \leq T,\ \zeta_0^\nu = \nu) \tag{5.3}$$

admits a solution such that for each $(t,\omega)\in[0,T]\times\Omega$, the map $\xi\to\chi^\xi_t$ (resp. $\nu\to\zeta^\nu_t$) is an increasing diffeomorphism of $\mathbb{R}$. The derivative $\partial_\xi\chi^\xi_t$ solves the SDE

$$d\partial_\xi\chi^\xi_t=\partial_2 a(t,\chi^\xi_t)\partial_\xi\chi^\xi_t dW_t+\partial_2 b(t,\chi^\xi_t)\partial_\xi\chi^\xi_t dt,\ \partial_\xi\chi^\xi_0=1.$$

Moreover

$$d\mathbb{P}\ a.s.,\ \forall(t,\xi)\in[0,T]\times\mathbb{R},\ \zeta_t^{\chi^\xi_T}=\chi^\xi_{T-t}. \tag{5.4}$$

Proof of Proposition 1.1 : Under our assumptions, the functions $x\sigma(t,x)$, $x\beta(t,x)$ and $x(\sigma\partial_2\eta-\beta)(t,x)$ vanish for $x=0$ and are Lipschitz continuous in x uniformly for $t\in[0,T]$. Therefore existence and trajectorial uniqueness follows from standard result concerning SDEs. For $[\bar a,\bar b](t,\xi)=\left[\sigma,\beta-\frac{\sigma^2}{2}\right](t,e^\xi)$, one has

$$[\bar a\partial_2\bar a-\bar b](t,\xi)=\left[\sigma e^\xi\partial_2\sigma+\sigma^2-\beta-\frac{\sigma^2}{2}\right](t,e^\xi)=\left[\sigma\partial_2\eta-\beta-\frac{\sigma^2}{2}\right](t,e^\xi). \tag{5.5}$$

By Itô's formula, one easily deduces that for $\bar\chi$ and $\bar\zeta$ solving

$$\begin{aligned}
d\bar\chi^\xi_t&=\bar a(t,\bar\chi^\xi_t)dW_t+\bar b(t,\bar\chi^\xi_t)dt+dL_t,\ t\le T,\ \chi^\xi_0=\xi\\
d\bar\zeta^\nu_t&=\bar a(T-t,\bar\zeta^\nu_t)d\hat W_t+\left(\bar a\partial_2\bar a-\bar b\right)(T-t,\bar\zeta^\nu_t)+d\hat L_t,\ t\le T,\ \bar\zeta^\nu_0=\nu.
\end{aligned}$$

$(e^{\bar\chi^{\log x}_t})_{t\in[0,T]}$ solves the SDE satisfied by X^x and $(e^{\bar\zeta^{\log z}_{t+}})_{t\in[0,T]}$ where by convention $\zeta^\nu_{T+}=\zeta^\nu_T$ solves the SDE satisfied by Z^z as soon as $e^{\zeta^{\log z}_{0+}}=z$. Since the last equality holds as soon as $L_{T^-}=L_T$ and therefore with probability 1, by trajectorial uniqueness,

$$\mathbb{P}\left(Z^z_T=e^{\zeta^{\log z}_T}\text{ and }\forall t\in[0,T],\ (X^x_t,Z^z_t)=(e^{\bar\chi^{\log x}_t},e^{\bar\zeta^{\log z}_{t+}})\right)=1.$$

With (5.5) one deduces (1.1).

For a fixed realization of μ or equivalently of the Lévy process $(L_t)_{t\in[0,T]}$, setting $[a,b](t,\xi)=[\bar a,\bar b](t,\xi+L_t)$, the process $\chi^\xi_t=\bar\chi^\xi_t-L_t$ and $\zeta^\nu_t=\bar\zeta^{\nu+L_T}_t-L_{T-t}$ respectively solve (5.2) and (5.3). Since the functions a and b satisfy the hypotheses in Proposition 5.1, this result implies that $\xi\to\chi^\xi_T=\bar\chi^\xi_T-L_T$ and $\nu\to\zeta^\nu_T=\bar\zeta^{\nu+L_T}_T$ are inverse increasing diffeomorphisms of $\mathbb{R}$ and

$$\partial_\xi\bar\chi^\xi_T=\partial_\xi\chi^\xi_T=e^{\int_0^T\partial_2 a(t,\chi^\xi_t)dW_t+\int_0^T(\partial_2 b-\frac12(\partial_2 a)^2)(t,\chi^\xi_t)dt}=e^{\int_0^T\partial_2\bar a(t,\bar\chi^\xi_t)dW_t+\int_0^T(\partial_2\bar b-\frac12(\partial_2\bar a)^2)(t,\bar\chi^\xi_t)dt}.$$

Since $\xi=\zeta_T^{\chi^\xi_T}=\bar\zeta_T^{(\bar\chi^\xi_T-L_T)+L_T}=\bar\zeta_T^{\bar\chi^\xi_T}$, $\xi\to\bar\chi^\xi_T$ and $\nu\to\bar\zeta^\nu_T$ and therefore $x\to X^x_T$ and $z\to Z^z_T$ are inverse increasing diffeomorphisms. Equality (1.3) follows easily. Moreover, on the almost sure event $\{\forall t\in[0,T],\ X^x_t=e^{\bar\chi^{\log x}_t}\}$,

$$\partial_x X^x_T=\frac1x\partial_\xi\bar\chi^{\log x}_T e^{\bar\chi^{\log x}_T}=e^{\int_0^T(\bar a+\partial_2\bar a)(t,\bar\chi^{\log x}_t)dW_t+\int_0^T[\bar b+\partial_2\bar b-\frac{(\partial_2\bar a)^2}{2}](t,\bar\chi^{\log x}_t)dt+L_T}.$$

One deduces (1.2) by remarking that $\bar a+\partial_2\bar a(t,\xi)=(\sigma+e^\xi\partial_2\sigma)(t,e^\xi)=\partial_2\eta(t,e^\xi)$ and

$$\begin{aligned}
\left[\bar b+\partial_2\bar b-\frac{(\partial_2\bar a)^2}{2}\right](t,\xi)&=\left[\beta-\frac{\sigma^2}{2}+e^\xi\partial_2\beta-e^\xi\sigma\partial_2\sigma-\frac{(e^\xi\partial_2\sigma)^2}{2}\right](t,e^\xi)\\
&=\left[\beta+e^\xi\partial_2\beta-\frac{(\sigma+e^\xi\partial_2\sigma)^2}{2}\right](t,e^\xi)=\left[\beta+e^\xi\partial_2\beta-\frac{(\partial_2\eta)^2}{2}\right](t,e^\xi).
\end{aligned}$$

$\blacksquare$

Proof of Proposition 1.2 : When for fixed $z > 0$, the function σ does not vanish in a neighbourhood of (T, z), then for some $\varepsilon \in (0, T)$ the function a does not vanish on $[T - \varepsilon, T] \times [\log z - 2\varepsilon, \log z + 2\varepsilon]$. Conditionally on $(L_t)_{t\in[0,T]}$, as soon as $L_{T^-} = L_T$, by Bouleau and Hirsch absolute continuity criterion (see Theorem 2.1.3 p.162 [3]) , $\bar{\zeta}_T^{\log z} = \zeta_T^{\log z - L_T}$ has a density with respect to the Lebesgue measure and for all $\xi \in \mathbb{R}$, either $\mathbb{P}(\bar{\chi}_T^\xi \in [\log z - \varepsilon, \log z + \varepsilon]) = \mathbb{P}(\chi_T^\xi \in [\log z - L_T - \varepsilon, \log z - L_T + \varepsilon]) = 0$ or the conditional law of $\bar{\chi}_T^\xi = \chi_T^\xi + L_T$ given $\chi_T^\xi \in [\log z - L_T - \varepsilon, \log z - L_T + \varepsilon]$ has a density. As $\mathbb{P}(L_{T^-} = L_T) = 1$, one deduces (1.4).
Let us now suppose that $\beta(t, x) = \gamma$. Then $X_t^x = xe^{\int_0^t \sigma(s,X_s^x)dW_s + \gamma t - \frac{1}{2}\int_0^t \sigma^2(s,X_s^x)ds + L_t}$ and

$$\mathbb{E}(e^{-\gamma t}X_t^x) = x\mathbb{E}\left(e^{L_t}\mathbb{E}\left(e^{\int_0^t \sigma(s,X_s^x)dW_s - \frac{1}{2}\int_0^t \sigma^2(s,X_s^x)ds}\middle|(L_s)_{s\in[0,T]}\right)\right).$$

Since σ is bounded, the conditional expectation in the right-hand-side is equal to 1 by Novikov's criterion (Proposition 1.15 p.308 [8]) and the right-hand-side is equal to x as $\mathbb{E}(e^{L_t}) = 1$ (take $u = -i$ in (0.3)). Moreover, as the function $\partial_2\eta(t, x) = (\sigma + x\partial_2\sigma)(t, x)$ is bounded, the expectation of $e^{-\gamma T}\partial_x X_T^x = e^{\int_0^T \partial_2\eta(s,X_s^x)dW_s - \frac{1}{2}\int_0^T (\partial_2\eta)^2(s,X_s^x)ds + L_T}$ is equal to 1 by the same argument.
Let $(h_n)_{n\geq 0}$ be a sequence of non-zero real numbers greater than $-x$ converging to zero. The random variables $\left((X_T^{x+h_n} - X_T^x)/h_n\right)_{n\geq 0}$ converge to $\partial_x X_T^x$ as $n \to +\infty$. Since they are non-negative and

$$\lim_{n\to+\infty} \mathbb{E}\left(\frac{X_T^{x+h_n} - X_T^x}{h_n}\right) = e^{\gamma T} = \mathbb{E}\left(\partial_x X_T^x\right),$$

they converge in L^1 to $\partial_x X_T^x$, which ensures uniform integrability. $\blacksquare$

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